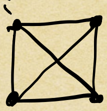
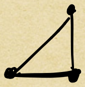
- Complete graph : every vertex is adjacent to every other vertex.
  A complete graph of n vertices is denoted K_n.

K_4:

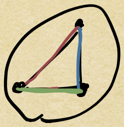


- Subgraph : a graph H is a subgraph of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$



- Clique : a complete subgraph

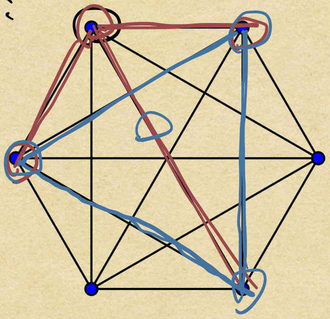
- Edge coloring : 2-coloring, 3-coloring, k-coloring, etc.



1953 Putnam Competition

Given a group of 6 people, show that at least 3 people are mutual friends OR at least 3 people are mutual strangers.

K_6 :



① color the edges using either blue or red.

② show that there must be a monochromatic triangle that is a subgraph.

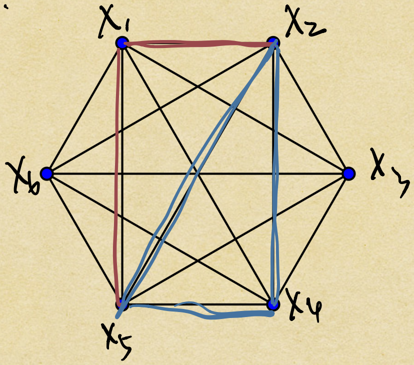
③ Pigeonhole principle.

Suppose $x_1, x_2, x_3, x_4, x_5, x_6$ are irrational numbers.

Prove that $\exists\, i, j, k$ s.t.

$x_i + x_j$, $x_j + x_k$, and $x_i + x_k$ are all irrational.

K_6 :



the sum is irrational

the sum is rational

$x_1 = -\pi$, $x_2 = \pi$

$x_1 + x_2 = 0 \in \mathbb{Q}$

$x_4 = \sqrt{2}$

$x_2 + x_4 = \pi + \sqrt{2} \in \mathbb{R} - \mathbb{Q}$

$x_1 + x_2 \in \mathbb{Q}$

$x_2 + x_5 \in \mathbb{Q}$

$\underline{x_1 + x_5} \in \mathbb{Q}$

$\underline{x_1 + x_5 + 2x_2} \in \mathbb{Q}$

$x_2 \in \mathbb{Q}$

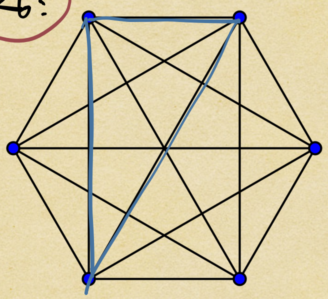
Ramsey Number

Definition:

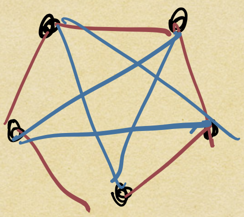
$\underline{R(s,t)} = n$:= the minimum n s.t. any 2-coloring on K_n must have either a clique (complete subgraph) of order s whose edges are monochromatic in color 1 **OR** a clique of order t whose edges are monochromatic in color 2.

e.g. $R(3,3) = n \leq 6$

K_6:

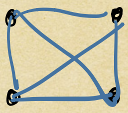


K_5:



> **Ramsey's Theorem:**
>
> $\forall s,t \in \mathbb{N}$, $s,t \geq 2$, $\underline{R(s,t)}$ is finite

$\underline{R(4,4)}$



$R(10,10) = n$ If we do 2-coloring on K_n, we are guaranteed to get a monochromatic complete subgraph of order 10 (K_{10}).

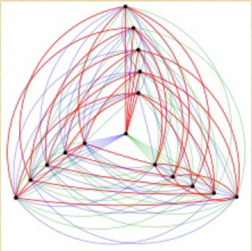
Generalized Ramsey's Theorem:

$\underbrace{R(n_1, n_2, \cdots, n_k)}_{k \text{ terms} \to k\text{-coloring}}$ is finite $\forall\, n_i \in \mathbb{N},\ n_i \geq 2$

$\underbrace{R(3,3,3)}_{k=3} = 17$

$R(n_1, n_2, \cdots, n_k) = \underline{m}$. $\exists\, m$ s.t. in a k-coloring of K_m, we are guaranteed to be able to find a monochromatic subgraph K_{n_i}.

K_{16}:



Schur's Theorem

$\forall\, k \geq 2,\ \exists\, n > 3$ s.t. given any k-coloring on the first n positive integers, there will be monochromatic x, y, and z s.t. $x + y = z$.

e.g. $k = 2,\ n = ?$ 2-coloring:

$N = \{\underline{1}, \underline{2}, \underline{3}, \cdots \underline{n}\}$

$\underline{R}: \{m \mid m \text{ is colored red}\} = \{1, n, \cdots\} \subseteq N$

$\underline{B}: \{m \mid m \text{ is colored blue}\} = \{2, 3, \cdots\} \subseteq N$

$x, y, z \in R$ OR $x, y, z \in B$
s.t. $x + y = z$.

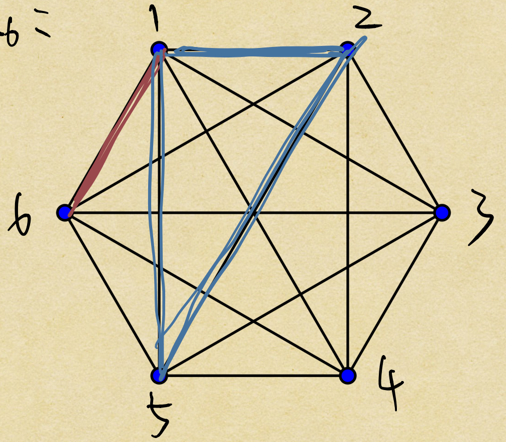
$k = 2$, $n = 5$ 2-coloring:

$N = \{1, 2, 3, 4, 5\}$

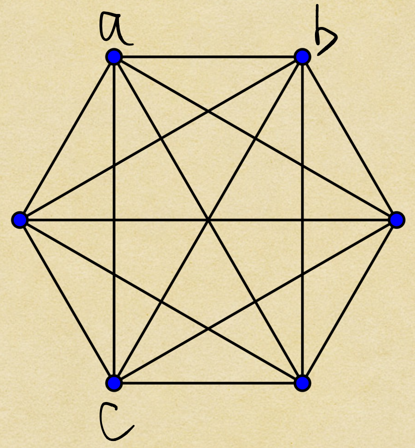
$x = 1, y = 3, z = 4$

Try: $n = 4$

$K_6 =$



$2 - 1 = \underline{1}$
$6 - 1 = 5$
$5 - 2 = 3$
$5 - 1 = 4$



$c > b > a$

let $\begin{cases} x = \underline{c - b} \\ y = \underline{b - a} \end{cases}$
$z = \underline{c - a}$

$x + y = c - a = z$

k-coloring: $R(n_1, n_2, \cdots, n_k) = m$